Matrix Metrics:  
Network-Based Systemic Risk Scoring 

Sanjiv Ranjan Das  
Leavey School of Business  
Santa Clara University  
Email: srdas@scu.edu  
http://algo.scu.edu/~sanjivdas/  

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Abstract

Matrix Metrics: Network-Based Systemic Risk Scoring

I propose a novel framework for network-based systemic risk measurement and management. I define a new systemic risk score that depends on the level of individual risk at each financial institution and the interconnectedness across institutions, and is generally applicable irrespective of how interconnectedness is defined. This risk metric is decomposable into risk contributions from each entity, forming a basis for taxing each entity appropriately. We may calculate risk increments to assess the potential risk of each entity on the overall financial system. The paper develops other subsidiary risk measures such as system fragility and entity criticality. An assessment using a measure of spillover risk is obtained to determine the scale of externalities that one bank might impose on the system; the metric is robust to this cross risk, and does not induce predatory spillovers. The analysis suggests that splitting up too-big-to-fail banks does not lower systemic risk.
1 Introduction

Morpheus: Unfortunately, no one can be told what the Matrix is.
You have to see it for yourself. —The Matrix

This paper proposes a new measure of aggregate systemic risk, and additional system-wide and entity-specific metrics, as a complement to existing measures of systemic risk. The measure provides a quantification of system-wide risk based on the level of vulnerability of each node in the system, and the interconnectedness of all nodes in the network (see Alter, Craig, and Raupach (2014) for an approach that also uses the same two quantities). This metric is easy to compute and has many appealing characteristics. Systemic risk (as opposed to systematic risk) has become an important concern after the financial crisis of 2008. Measuring this risk and managing it are two salient goals of the analysis in this paper.

Systemic risk is not always easy to define. But there exist some universally accepted characteristics in the extant literature (discussed below): a risk that has (a) large impact, (b) is widespread, i.e., affects a large number of entities or institutions, and (c) has a ripple effect that endangers the existence of the financial system. The mortgage/financial crisis of 2008 certainly had all these three characteristics, but the market crash of 1987 impacted only a small set of assets (equities) and did not endanger the financial system. However, definitions of systemic risk abound and economists may not agree on any single one. We describe and discuss some popular measures that are related to our new measure.

There is a growing literature on systemic risk measurement in finance, and we mention some representative papers here, though there is a range of papers similar to these. Much of this literature uses equity returns of financial institutions and the correlations of these returns to construct systemic risk measures. An important paper is Billio, Getmansky, Lo, and Pelizzon (2012); they use return correlations and Granger causality regressions on returns to construct network maps and develop network measures of systemic risk. Joint extreme tail risk is also used as a systemic risk measure, such as the well-known CoVaR metric of Adrian and Brunnermeier (2010). The SES (systemic expected shortfall) measure of Acharya, Pedersen, Philippon, and Richardson (2011) examines tail risk for a financial institution when the aggregate system is under stress. This is similar to the DIP (distressed insurance premium) metric of Huang, Zhou, and Zhou (2011). Kritzman, Li, Page, and Rigobon (2011) develop the Absorption Ratio (AR), based on when the comovement of returns of assets in a principal components analysis becomes concentrated in a single factor. A modification of this approach by Reyngold, Shnyra, and Stein (2013) denoted Credit Absorption Ratio (CAR) extends AR to default risk data. And Carciente, Kenett, Avakia, Stanley, and Havlin (2015) develop a methodology for stress testing banks for systemic risk,
using a bipartite graph of financial institutions and assets.

The systemic risk measure in this paper is different from the ideas in these related papers. First, it does not depend only on equity returns, as it is general and can be used with any measure of interconnectedness. For example, a network graph generated from interbank transactions (FX, CDS, loans, etc.) may be used as developed in Burdick et al. (2011), as might be the network generated from Granger causality analysis in Billio, Getmansky, Lo, and Pelizzon (2012); and Billio, Getmansky, Gray, Lo, Merton, and Pelizzon (2014). Second, the measure separates two aspects of overall risk: compromise level (i.e., risk score at each node) and connectivity (i.e., the network graph) across nodes; it explicitly uses the network configuration in scoring systemic risk. Third, an important property of this aggregate systemic risk measure is that it is decomposable additively into individual contributions to systemic risk, enabling imposing a tax on financial institutions, if a regulator so chooses, for individual institutional contributions to aggregate risk. A similar suggestion is made in Acharya, Pedersen, Philippon, and Richardson (2011).

In addition to these features of the systemic risk score, other useful attributes and applications of this measure are as follows. One, it may be used in combination with network centrality scores to manage the risk of the financial system. The criticality of a node in the financial system is defined as the product of its risk (compromise) level and its centrality. Two, we propose a measure of fragility that is related to concentration risk, i.e., resembles a Herfindahl-Hirschman index. This enables assessment of the speed with which contagion can spread in the system. Three, we compute risk increments of the aggregate systemic risk score, i.e., the extent to which each node in the system contributes to aggregate risk if its level of compromise increases by unit amount. This enables identifying which nodes are critical, even though they may not be compromised at the current time. Fourth, we define a normalized systemic risk score as well, which quantifies the network effect present in the system. This complements the fragility score.

In addition to these stand-alone static metrics, we explore a few comparative statics in order to understand the dynamics of the network, without a full-blown dynamic analysis. First, we examine cross risk, i.e., the externality effect of one node’s increase in risk on the risk contribution of other nodes. We explore this risk numerically and find that cross risk is low, i.e., it is not easy for a badly performing node to impose large externalities on the other nodes in terms of our metric, making it robust for practical use. Second, we examine whether breaking large banks into smaller banks helps reduce systemic risk, and find that this remedy does not work. Instead, eliminating too-big-to-fail banks exacerbates systemic risk as it increases points of failure in the system.
This paper proceeds as follows. In Section 2 we present the notation and structure of the new systemic risk score, and we also present related network measures. In Section 3 we extend the measure to a normalized one, and provide more examples. In order to set this metric in context, we provide discussion in Section 5 that compares the new metric to other systemic risk measures, summarized in Appendix. Section 6 provides brief concluding discussion.

2 Modeling

2.1 Notation

Risk in a connected network arises from compromised nodes and their connections. We propose and define a parsimonious and intuitive metric for quantifying the aggregate risk in a network of related entities and explore its properties.

Assume that the network comprises $n$ nodes and is formally defined as a graph $G(V,E)$ where $V \in \mathcal{R}^n$ is the vertex (node) set of entities or banks, and $E \in \mathcal{R}^{n \times n}$ is the edge (link) set comprising elements $E(V_i, V_j) \equiv E_{ij} \in \{0, 1\}$, denoting which nodes are connected to each other. The graph may be assumed to be directed, i.e., $E_{ij} \neq E_{ji}$, and undirected graphs are special cases. Also, $E_{ii} = 1, \forall i$, and we will see that this is needed for computing the risk score below. The link $E_{ij}$ in this network is to be interpreted as a flow/effect from node $i$ to $j$ in the sense that if bank $i$ is impacted economically, then it will transmit this impact to bank $j$.

See Exhibit 1 for an example of the network and matrix. The network is represented by an $(n \times n)$ adjacency matrix with all elements in $\{0, 1\}$. However, one may imagine more complex networks where the connectivity is not binary, but depends on the degree of interaction between nodes. These matrices may be normalized such that the diagonal $E_{ii} = 1, \forall i$, and the off-diagonal elements are scaled to values $E_{ij} / \text{max}(E_{ij}), \forall i, j, i \neq j$, thereby extending the adjacency matrix from the binary $\{0, 1\}$ case to values in the range $[0, 1]$. Higher values would denote a connection with greater influence. The set up is simple, yet general.

For each node $V_i$ we define the level of compromise as $C_i \geq 0$. The risk vector for all nodes is $C = [C_1, C_2, ..., C_n]^\top \in \mathcal{R}^n$. Our risk score is agnostic as to how compromise is defined. For example, a good measure of compromise to use would be the Altman (1968) Z-score. Another choice would be the expected loss measure for a financial institution as used

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1The networks in this paper are not required to be symmetric ($E_{ij} = E_{ji}$) or regular ($\sum_{i \neq j} E_{ij} = \sum_{j \neq i} E_{ji} = \text{constant}$) as defined in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013).
in Acharya, Pedersen, Philippon, and Richardson (2011). We may also use credit spreads or credit ratings.

In this formulation, there is no notion of the relative size of the nodes. For example, a small hedge fund could have the same credit score as a low-rated very large investment bank. However, this is not a severe limitation because the investment bank might have a greater influence on other banks than the hedge fund, either through a greater number of links in the network or via stronger links, using the more generalized version of the E matrix, where values $E_{ij} \in [0, 1]$.

### 2.2 Systemic Risk Score (S)

We now define a single systemic risk score for the aggregate system, that accounts for the connections between institutions and the level of individual compromise at each node in the network. This is the main measure developed in this paper. (Bold font in the equations represents either a vector or matrix)

**Definition:** The risk score for the entire network is

$$S(C, E) = \sqrt{C^\top E C}$$

Scalar $S$ is a function of the compromise level vector $C$ for all nodes and the connections between nodes, given by adjacency matrix $E$.

The function $S(C, E)$ has some useful mathematical properties. First, it is linear homogenous in $C$, and this will be shown to be useful in the ensuing analytics where we need to decompose the aggregate risk score into contributions from each node. Second, as long as all numbers in the $C$ vector and $E$ matrix are positive, the value of $S$ remains positive as well. Third, the metric is analogous to portfolio return standard deviation, where we have replaced portfolio weights with $C$ and the covariance matrix of returns with $E$. The pre- and post-multiplication of the adjacency matrix $E$ with the credit score vector $C$ ensures that we obtain a scalar quantity, and that it is also linear homogenous in $C$. The difference between $E$ and the covariance matrix in the standard portfolio problem is that $E$ is not symmetric, though it is positive definite.

This measure is heuristic, but the economic motivation for the metric comes from two important economic underpinnings of systemic risk. This risk is characterized by (a) the interconnectedness of financial institutions (nodes), and (b) the credit quality of these nodes, as suggested in Billio, Getmansky, Lo, and Pelizzon (2012). To cite them - “From a theoretical
perspective, it is now well established that the likelihood of major financial dislocation is related to the degree of correlation among the holdings of financial institutions, how sensitive they are to changes in market prices and economic conditions (and the directionality, if any, of those sensitivities, i.e., causality), how concentrated the risks are among those financial institutions, and how closely linked they are with each other and the rest of the economy."

Example: Suppose we have 18 nodes in a network, depicted by the adjacency matrix, i.e., the directed, unweighted graph shown in Exhibit 1. The compromise vector is \( C = [0, 0, 1, 2, 2, 2, 2, 1, 0, 2, 2, 2, 1, 0, 1, 1] \), where 0 is no compromise, 1 is a low level of compromise, and 2 indicates a highly compromised node. We may think of these values as credit rating scores, where the higher the score, the worse the credit quality of the financial institution. The risk score using equation (1) is \( S = 11.62 \). This is interpretable as the systemic credit score of the financial system. The value per se does not connote any meaning, whereas the changes in systemic score \( S \) are what may be tracked by a regulator over time. When there is a sudden spike in \( S \), investigation is made as to which financial institution is most responsible for incrementing systemic risk, which we compute using a risk decomposition metric, developed in Section 2.4. Before turning to this, we present in the next section an older, very useful metric for the importance of a node, denoted “centrality”.

2.3 Centrality \( (x) \) and Criticality \( (y) \)

Whereas the main metric developed in this paper is \( S \), understanding systemic risk, and drilling down into its constituents leads us to many other measures that relate to attributes of systemic risk. In this and subsequent sections, we explore these other measures as well.

We often wish to know which node in the network is most central, i.e., has the most important location in terms of being connected to other nodes in the network. In a complicated network, this is often not such an easy question to answer. Here we employ the well-known notion of “centrality” developed by sociologists a few decades ago. We note that the influence of any node in a network (denoted \( x_i \)) comes from which other nodes \( j \) it is connected to (denoted by the edges in the adjacency matrix, \( E_{ij} \)), and in turn these nodes are impacted by the nodes they are connected to and so on. This circularity may be

\[ \text{This is a static model, and at any point in time the credit scores for each bank are a given quantity. No doubt, to the extent that banks fortunes are correlated based on common economic factors, these credit scores are likely to reflect that. But in a static model we do not require dynamics with an underlying correlation of credit scores. Also, the network adjacency matrix will capture some of the connections between banks' fortunes, and so correlation of credit quality may be implicit, despite no explicit modeling of correlations.} \]
Exhibit 1: Directed network of 18 nodes. One-way arrows means that risk flows in the direction of the arrow. Two-way arrows mean risk flows in both directions. The network is summarized in the adjacency matrix. Note that the diagonal values are all 1. The “diameter” of this network, i.e., the maximal shortest distance between any two nodes is 2.
Exhibit 2: **Centrality**: Normalized centrality for each node in the network shown in Exhibit 1, rank ordered for display.

represented in the following system of $n$ equations:

$$x_i = \sum_{j=1}^{n} E_{ij}x_j, \quad \forall i = 1, \ldots, n$$

The left-hand side of this system of equations is a $n$-vector $x$ which provides a score for the influence or centrality of each node in the network. This leads to the following definition of centrality.

**Definition**: Eigenvalue centrality is the normalized principal eigenvector $x \in \mathbb{R}^n$, that for scalar $\lambda$, satisfies the eigensystem

$$\lambda \ x = \mathbf{E} \ x \quad (2)$$

Centrality was first defined in Bonacich (1987), and popularized more recently as the PageRank algorithm by Google [Brin and Page (1998)].

**Example**: We compute centrality for this network and plot it in Exhibit 2.

Centrality is normalized where the highest centrality node is set to value 1, and the other node values are relative centrality to this node. Neither centrality or fragility depend on the compromise vector $\mathbf{C}$ since it is computed only using adjacency matrix $\mathbf{E}$. In order to expand the concept of centrality to accounting also for the compromise levels at each node, we define a new metric called “criticality”. 
Exhibit 3: Criticality: Criticality for each node in the network shown in Exhibit 1 rank ordered for display.

Definition: Criticality is compromise-weighted centrality. This new measure is defined as $y = C \times x$ where $y, C, x \in \mathbb{R}^n$. Note that this is an element-wise multiplication of vectors $C$ and $x$.

Critical nodes need immediate attention, either because they are heavily compromised or they are of high centrality, or both. It offers a way for regulators to prioritize their attention to critical financial institutions, and pre-empt systemic risk from blowing up. We compute criticality for this network and plot it in Exhibit 3.

The node numbers in Exhibits 2 and 3 are the same nodes in our continuing example. These would be financial institutions in application to an economic system. Note that the centrality scores in Exhibit 2 are ordered differently than the criticality scores in Exhibit 3. This is because centrality ordering does not depend on the credit quality of the banks ($C$). Hence, node 1, which is the most connected is the one with the highest centrality, and node 5 has low centrality; for a visual feel see Exhibit 1. However, criticality depends on both centrality and credit quality, and so node 1 has very low criticality as this bank has a high credit quality, whereas node 5 now has high criticality. Some nodes such as 11 and 12, have moderate centrality and credit quality and hence remain high in both Exhibits, on metrics of centrality and criticality.
2.4 Risk Decomposition (D)

We exploit the linear homogeneity of the function $S(C, E)$ in $C$ using Euler’s equation that decomposes first-order homogenous functions, resulting in a representation of the aggregate systemic risk score into node-wise components.

**Definition:** Risk Decomposition is the attribution of the aggregate network risk score $S$ to each node’s individual risk contribution $D_i, i = 1, 2, ..., n$, such that $S = \sum_{i=1}^{n} D_i$.

The risk contribution of each node is $D_i = \frac{\partial S}{\partial C_i} C_i$.

This decomposition formula is the result of applying Euler’s theorem\(^3\) to the function $S(C, E)$ and provides a decomposition of the system-wide risk score $S$ into the contribution of each node to total risk, as well as shows that the individual risk contributions sum up to the total systemic score $S$.

\[
S = \frac{\partial S}{\partial C_1} C_1 + \frac{\partial S}{\partial C_2} C_2 + \ldots + \frac{\partial S}{\partial C_n} C_n \tag{3}
\]

When a node fails and exits the network, the systemic risk score for the network, and the risk contribution of each node within the network will change as well. Ceteris paribus, removal of a node will lower systemic risk. The effect is analogous to quarantining a node. But when a node fails it may also impact the credit quality of other nodes, i.e., some $C_i$ will worsen, even though the “size” of the adjacency matrix $E$ becomes smaller as some connections are removed. The overall effect on the systemic risk score $S$, and risk contributions $D_i$ is therefore indeterminate. In order to establish an expected change in these risk scores, the model here needs to be extended from a static model to a dynamic one.

**Example:** We compute the risk decomposition of the network in Exhibit 1 and this is shown in Exhibit 4 where $\sum_{i=1}^{n} D_i = 11.62$. Note that the numbers $D_i$ for each node $i$ depend on both the compromise vector $C$ and the network adjacency matrix $E$. In this risk network, nodes 5 and 8 contribute the most to system-wide risk. We note that even though these nodes are not central in the network, they have a high level of compromise $C_i, i = 5, 8$ and therefore are the ones to be monitored most.

This risk decomposition is especially useful for pinpointing the network effect when there is a sudden rise in systemic risk score $S$. By examining the changes in risk contribution for each node from one period to the next, critical causal nodes are quickly identified. Further, one may determine whether the increase in a node’s risk contribution arises from an increase in its compromise level or from an increase in its connectivity.

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\(^3\)Euler’s theorem states that if a function $f(x), x \in \mathbb{R}^n$ is homogenous of degree 1, then it may be written as $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i$. 
Exhibit 4: **Risk Decomposition**: The risk contribution $D_i$ for each node in the network shown in Exhibit [1] rank ordered for display. The aggregate risk is $\sum_{i=1}^{20} D_i = 11.62$.

### 2.5 Risk Increment ($I$)

A regulator may be interested in assessing how a node in the financial network is likely to impact the system should that node become excessively compromised. This is determined by computing risk increments.

**Definition**: Risk Increment is the change in the aggregate network risk score $S$ when the compromise score $C_i$ of an asset changes, i.e., $I_i = \frac{\partial S}{\partial C_i}$.

Given $S = [C^\top E C]^{1/2}$ the derivative with respect to $C$ is the vector

$$I = \frac{\partial S}{\partial C} = \frac{1}{2S}[EC + E^\top C] \in \mathbb{R}^n$$

which is easy to compute even for large $n$.

**Example**: We compute the risk increments of the network in Exhibit [1] and this is shown in Exhibit [5]. Note that the numbers $I_i$ for each node $i$ depend on both the compromise vector $C$ and the network adjacency matrix $E$.

Note that node 1 has a very low current risk contribution (as shown in Exhibit [4]), but has the potential to be very risky as it has the highest risk increment (see Exhibit [5]), emanating from the fact that it is a highly connected node.

Both, risk contribution and risk increment are useful in identifying the source of system vulnerabilities, and in remediation. In assessing whether a node should be allowed to fail,
Exhibit 5: *Risk Increment* \( I_i \) for each node in the network shown in Exhibit 1, rank ordered for display.

we may disconnect it from the network and assess how these metrics are impacted.

### 2.6 Fragility (\( R \))

A more concentrated network is one where a few nodes have many connections whereas most nodes have very few. A highly concentrated network tends to have greater risk transmission because once a highly central node is compromised, the malaise rapidly spreads to many other nodes. This propensity for risk to spread on a network is denoted as “fragility.”

**Definition:** Let \( d \) be the degree of a node, i.e., the number of connections it has to other nodes. Define the fragility of the network to be

\[
R = \frac{\mathbb{E}(d^2)}{\mathbb{E}(d)}
\]

where the function \( \mathbb{E}(\cdot) \) stands for the expectation of the random variable in the function.

Keeping \( \mathbb{E}(d) \) constant, an increase in concentration results in an increase in \( \mathbb{E}(d^2) \), with a corresponding increase in fragility \( R \). This definition is intuitive, and the fragility measure is similar to a normalized Herfindahl-Hirschman index (which is the numerator). If the network’s connections are concentrated in a few nodes, we get a hub-and-spoke network (also known as a scale-free network) on which spread of a shock is rapid, because once a
node with many connections is infected, disease on a network spreads rapidly.

Consider for example a network with four nodes each with degree 2, a network that is not fragile, i.e., fragility score is low, \( R = 2 \), but the same network of four nodes with degrees \{4, 2, 1, 1\} has the same mean degree, but is much more fragile as \( R = 11 \). Concentration of degree induces fragility. This metric is a useful complement to the systemic risk score \( S \). The fragility of the example network used in this paper is computed to be 7.94.

One may simply wish to look at the Herfindahl index \( \mathcal{E}(d^2) \), which is often used, but in this case, the normalization by \( \mathcal{E}(d) \) is relevant because it ensures that a smaller more concentrated network with fewer connections is more fragile than a network with many connections but less concentration. For example, consider the following two networks shown below.

Network A is the typically fragile hub and spoke network, and network B is less fragile as it does not have this structure. Yet if we just computed the Herfindahl index \( \mathcal{E}(d^2) \) it would have values 7 and 7.67, respectively, for A and B, indicating that network B is more fragile. This is because B has more overall degree. Therefore, we normalize by \( \mathcal{E}(d) \), which is 2.14 and 2.67, respectively. After normalization, fragility is higher for network A, equal to 3, whereas for B is it equal to 2.67.

Finally, while fragility is a measure for the entire network, centrality is a measure for each node. Hence, they are different. There is a link between the two, because a network that has concentrated centrality in a few nodes, will likely be more fragile.

3 Extended Metrics

The previous section introduced several new network-based systemic risk measures such as the aggregate systemic risk score, risk decomposition, risk increment, fragility, and criticality. In this section, we modify and extend these metrics further.
3.1 Normalized Risk Score ($\bar{S}$)

The units of systemic risk score $S$ are determined by the units of compromise vector $C$. If $C$ is a rating, then systemic risk $S$ is measured in rating units. If $C$ is a Z-score (for instance), then $S$ is a system-wide Z-score. And if $C$ is expected loss, then $S$ is in system-wide expected loss units.

In order to compare the network effect across systems, we extend the score $S$ to normalized score $\bar{S}$:

$$\bar{S} = \frac{\sqrt{C^\top E C}}{\|C\|} = \frac{S}{\|C\|}$$  \hspace{1cm} (4)

where $\|C\| = \sqrt{C^\top C}$ is the norm of vector $C$. When there are no network effects, $E = I$, the identity matrix, and $\bar{S} = 1$, i.e., the normalized baseline risk level with no network (system-wide) effects is unity. We can use this normalized score to order systems by systemic risk. For the system in our example, the normalized score is $\bar{S} = 1.81$.

We note that this normalized measure may mask high levels of risk (i.e., if all firms were rated CCC). It is always better for a regulator to only look at $S$ and not $\bar{S}$. Therefore, this measure is useful in separating out the network effect, but is not to be used for measuring overall systemic risk.

3.2 Varying Risk or Connectivity

The addition of a link in the network will increase both $S$ and $\bar{S}$, ceteris paribus. And a reallocation of risk among nodes in vector $C$ will also change $S,\bar{S}$. Limiting or setting constraints on entries in matrix $E$ is akin to controls on counterparty risk in an interbank system, and limiting each entry in vector $C$ constrains own risk. A network regulator may choose limits in different ways to manage systemic risk. Simulating changes to $C$ and $E$ allows for generating interesting test case scenarios of systemic risk.

Example: (Increasing risk at a node) If we keep the example network unchanged, but reallocate the compromise vector by reducing the risk of node 3 by 1, and increasing that of node 16 by 1, we find that the risk score $S$ goes from 11.62 to 11.87, and the normalized risk score $\bar{S}$ goes from 1.81 to 1.85. This is because node 16 is marginally more central than node 3 (as may be seen from Exhibit 2).

In this manner, we may examine how adding a link to the network or removing a link may help in reducing system-wide risk. Or we may examine how additional risk at any node leads to more systemic risk. A system regulator can run these analyses to determine the best way to keep system-wide risk in check.
3.3 Cross/Spillover Risk ($\Delta D_{ij}$)

An increase in the risk level at any node $i$ does not only impact its own risk contribution $D_i$, but that of other nodes ($D_j, j \neq i$) as well through the network matrix. A single financial institution mismanaging its own risk might impose severe externalities in terms of potential risk on other banks in the system through network effects. In a situation where banks are taxed for their systemic risk contributions, for example, required to keep additional capital based on their individual risk contributions ($D_i$), externalities may instigate retaliatory actions that result in escalation in systemic score $S$. Hence, it is important to compute how severe cross risk might be. Espinosa-Vega and Sole (2010) point out that spillover risk is an important motivation for proposed capital surcharges for systemic risk in their model of financial surveillance.

We analyzed our sample network by computing the effect on risk contribution of each node if any other node has a unit increase in compromise level. We denote the cross risk of node $i$ when node $j$ has a unit increase in compromise level $C_j$ as $\Delta D_{ij} = \frac{\partial D_i}{\partial C_j}$, keeping the network topology $E$ constant. The results are shown in Exhibit 6. It is apparent that “cross” risk is insignificant compared to “own” risk contribution. This suggests that regulators need not be overly concerned with moral hazard on networks, where one node can impose severe externalities on other nodes. It also means that the risk metric $S$ is one that is not easily gamed for externalities, i.e., if institutions are taxed based on their risk contributions, then any single institution cannot impact the taxes of another in a material way. To this extent, the measure is robust.

The analysis of cross risk assumes that network adjacency matrix $E$ does not change with $C$. It is hard to say in which way network topology will change. It may be that worsening credit quality of a given bank reduces the number of connections, as some banks stop trading with it. This would reduce systemic risk. On the other hand, it may reallocate trading in a manner where trading becomes more concentrated in a few nodes, raising fragility and possibly systemic risk.

4 Risk Scaling and Real-World Application

We assess three questions here, in order to derive a deeper understanding of the properties of systemic risk score $S$. These questions pertain to how the score changes when we scale the level of compromise, the level of interconnectedness, and the breaking down of nodes into less connected ones. We also provide a summary of the application of the model to real world data.
Exhibit 6: Change in risk contribution when any node experiences a unit increase in compromise level. The impact from each node on every other node is shown. The upper plot is in bar form, and the lower plot is a heat map.
First, ceteris paribus, how does an across-the-board change in compromise vector $C$ impact $S$? The answer is simple – since $S$ is linear homogenous in $C$, this effect is purely linear.

Second, how does an increase in connectivity impact systemic risk $S$? Is this a linear or non-linear effect? We ran a simulation of a fifty node network and examined $S$ as the number of connections per node was increased. Simulation results are shown in Exhibit 7. The plot shows how the risk score increases as the probability of two nodes being bilaterally connected increases from 5% to 50%. For each level of bilateral probability a random directed network is generated of 50 nodes. We then set the diagonal to 1 as required. The rest of the off-diagonal elements are 1 or 0 and were generated by the random graph function. This is the simulated $E$ matrix. A compromise vector $C$ is also generated with equally likely values $\{0, 1, 2\}$. Using $C$ and $E$, we compute the systemic risk score $S$. This is repeated 100 times and the mean risk score across 100 simulations is plotted on the y-axis against the bilateral probability on the x-axis. These results based on random graph generation show that the risk score increases with connectivity, as expected, but in less than linear fashion (the plot is mildly concave). This relates to results in [Vivier-Lirimont (2006); Blume, Easley, Kleinberg, Kleinberg, and Tardos (2011); Gai, Haldane, and Kapadia (2011)] who show that dense interconnections destabilize networks as risk increases with increasing density. In our case, our systemic risk score also increases but less than linearly with the number of connections.

Third, we examine whether partitioning nodes into more numerous smaller entities reduces systemic risk (a question also addressed in very different models by [Cabrales, Gottardi, and Vega-Redondo (2014); Vivier-Lirimont (2006)]). The idea here is to assess whether breaking up too-big-to-fail banks into smaller entities will result in a reduction in systemic risk. Whereas the first two questions did not consider increasing or decreasing the number of nodes, in this case we explicitly increase the numbers of nodes while adjusting the average number of connections per node down, so as to keep the overall connectivity unchanged, while changing the structure of the network. The program logic is very much the same as in the previous simulation except that the $C$ and $E$ matrix are constructed differently, and the x-axis in the plots is the number of nodes in the network. Exhibit 8 shows that the risk score $S$ in fact increases! Hence, splitting large banks into smaller banks does not reduce systemic risk. Risk increases because the number of points of cascading failure increase when a large

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4We used the R programming language and the package igraph for these analyses. Random networks are generated using the function erdos.renyi.game. Another more complex approach is to use the Law of Preferential Attachment to generate only scale-free networks, though the results are likely to be the same, as we are exploring the density of the network rather than its structure.

5The shape of the plot in Exhibit 7 is unsurprising in retrospect, as the metric $S$ contains the adjacency matrix $E$ under the square root sign, and as $E$ gets denser, $S$ will look like a plot of the square root of increasing numbers, in mildly concave shape.
Exhibit 7: The increase in risk score $S$ as the number of connections per node increases. The plot shows how the risk score increases as the probability of two nodes being bilaterally connected increases from 5% to 50%. For each level of bilateral probability a random network is generated for 50 nodes. A compromise vector is also generated with equally likely values $\{0, 1, 2\}$. This is repeated 100 times and the mean risk score across 100 simulations is plotted on the y-axis against the bilateral probability on the x-axis.
Exhibit 8: The change in risk score $S$ as the number of nodes increases, while keeping the average number of connections between nodes constant. This mimics the case where banks are divided into smaller banks, each of which then contains part of the transacting volume of the previous bank. The plot shows how the risk score increases as the number of nodes increases from 10 to 100, while the expected number of total edges in the network remains the same. A compromise vector is also generated with equally likely values $\{0, 1, 2\}$. This is repeated 5000 times for each fixed number of nodes and the mean risk score across 5000 simulations is plotted on the y-axis against the number of nodes on the x-axis.

I would caution the reader to take this result on splitting banks up with a grain of salt. The fact that splitting banks into smaller ones increases $S$, keeping the number of connections overall constant, is a reduced-form result to evaluating a possible policy prescription to remedy the too-big-to-fail problem. It may be too simplistic an approach for the analysis of what is clearly a highly complex and controversial issue. It lacks a notion of size and implies that splitting small banks into even smaller ones will raise systemic risk, which may not be the case. Does this mean it’s best to collapse all banks into one large global bank? Obviously not, as other factors come into play, and the ceteris paribus nature of this analysis where we assume that credit quality remains same is not valid in the extreme. The outcome of this simple analysis here seems related to the network result where adding an extra road to a
network to relieve traffic congestion has the effect of increasing congestion for all. Therefore, despite its simplicity and myriad assumptions, this result does offer one starting point for the analysis of policies around too-big-to-fail banks.

Fourth, the program code for systemic risk networks was applied to real-world data in India to produce daily maps of the Indian banking network, as well as the corresponding risk scores. The credit risk vector $C$ was based on credit ratings for Indian financial institutions (FIs). The network adjacency matrix was constructed using the ideas in a paper by Billio, Getmansky, Lo, and Pelizzon (2012) who create a network using Granger causality. This directed network comprises an adjacency matrix of values $(0, 1)$ where node $i$ connects to node $j$ if the returns of bank $i$ Granger cause those of bank $j$, i.e., edge $E_{ij} = 1$. This was applied to U.S. financial institution stock return data, and in a follow-up paper, to CDS spread data from U.S., Europe, and Japan (see Billio, Getmansky, Gray, Lo, Merton, and Pelizzon (2014)), where the global financial system is also found to be highly interconnected.

In the application of the methodology of this paper to India, the network matrix is created by applying this Granger causality method to Indian FI stock returns.

The system is available in real time and may be accessed directly through a browser. To begin, different selections may be made of a subset of FIs for analysis. See Exhibit 9 for the screenshots of this step. Once these selections are made and the “Submit” button is hit, the system generates the network and the various risk metrics, shown in Exhibits 10 and 11, respectively.

5 Discussion of Other Measures of Systemic Risk

As a practical matter, several measures of systemic risk have been proposed, and each one implicitly defines systemic risk as that risk being quantitatively determined by their measure. This is definition by quantification, measurement as one sees it. In our setting of risk networks the system-wide risk scores $\{S, \bar{S}\}$ capture systemic risk as a function of the compromise vector $C$ and the network of connected risk entities $E$. Other research conducts this differently. Some measures of systemic risk are network-based but most of the measures are based on stock return correlations.

The Appendix provides a brief summary of some of the popular systemic risk measures proposed in the literature. These are the Granger causality based network of Billio, Getmansky, Lo, and Pelizzon (2012), the CoVaR metric of Adrian and Brunnermeier (2010), the Absorption Ratio of Kritzman, Li, Page, and Rigobon (2011), the Credit Absorption

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6Thanks to the Reserve Bank of India for sponsorship, and to InnovAccer (www.innovaccer.com) for collecting the data and hosting the site that runs the program code.
5 Discussion of Other Measures of Systemic Risk

Exhibit 9: Screens for selecting the relevant set of Indian FIs to construct the banking network. The first selection shows selecting only banks. The second selection selects within banks and we select all of them. The third panel shows the date of the selection.
Exhibit 10: Screens for the Indian FIs banking network. The upper plot shows the entire network as of December 3, 2015. The lower plot shows the network when we mouse over the bank in the middle of the plot. Red lines show that the bank is impacted by the other banks, and blue lines depict that the bank impacts the others, in a Granger causal manner.
Exhibit 11: Screens for systemic risk metrics of the Indian FIs banking network. The top plot shows the current risk metrics, and the bottom plot shows the history from 2008.
Ratio (CAR) measure of Reyngold, Shyra, and Stein (2013), and the Systemic Expected Shortfall measure of Acharya, Pedersen, Philippon, and Richardson (2011).

There is an important difference between the Granger causality based network of Billio, Getmansky, Lo, and Pelizzon (2012), the AR of Kritzman, Li, Page, and Rigobon (2011), and \( \text{CoVaR} \), versus the \( \text{SES} \) measure. The three former measures assess the impact a single bank has on the system, whereas the latter measure assesses the reverse, i.e., the impact of system-wide risk on each bank. The new measures of system-wide risk \( \{S, \bar{S}\} \) proposed in this paper are akin to the first approach, and I believe that this is the more relevant view of systemic risk, and offers an aggregate risk score as well. However, both approaches are relevant in computing extra systemic risk capital requirements.

There are some salient differences between these measures of systemic risk and the network score in this paper. First, these measures focus on the effect of failure of a given institution on others. Hence, they are pairwise and conditional. In contrast, network risk scores are system-wide and unconditional. Second, the measures are based on correlations, and correlations tend to be high in crisis periods but are not empirically established as early-warning indicators of systemic risk. Relying on stock return correlations as an early warning indicator of network risk is likely to be futile, as correlation matrices reflect systemic risk after the risk has arisen, rather than before. Network-based measures may be better at identifying if there is a systemic vulnerability prior to a system shock, but these too, need empirical validation. Third, correlation based measures tend to be removed from the underlying mechanics of the system, and are in the nature of implicit statistical metrics. Network-based measures directly model the underlying mechanics of the system because the adjacency matrix \( \mathbf{E} \) can be developed based on physical transaction activity. Further, the compromise vector is a function of firm quality that may be measured in multidimensional ways. This separation of network effect (connectivity) and individual bank risk (compromise), and their combination into a single aggregate risk score, offers a simple, practical, and general approach to measuring systemic risk.

This paper is not only related to the growing literature on measures of systemic risk, but also to the network literature in economics in papers like Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013); Allen and Gale (2000); Allen, Babus, and Carletti (2012), and the literature on risk in clearing systems, see Eisenberg and Noe (2001); Duffie and Zhu (2011), Borovkova and El Mouttalibi (2013). Systemic risk measures based on dynamic conditional correlations are also proposed, see Brownlees and Engle (2010); Engle, Jondeau, and Rockinger (2012).

Therefore, the novel framework in this paper may be used as a complement to existing approaches. Whether or not the network is derived from physical deal flow or from returns
data, the risk score $S$ may be computed, decomposed by node, and risk increments derived therefrom, along with many other metrics, to provide a useful dashboard for managing systemic risk.

## 6 Concluding Comments

This framework for network-based systemic risk modeling develops system-wide risk scores such as a new aggregate systemic risk score ($S$), a normalized score ($\bar{S}$), a fragility score ($R$), and also entity-specific risk scores: a risk decomposition ($D_i$), risk increments ($I_i$), criticality ($y_i$), and a score for spillover risk ($\Delta D_{ij}$). All these metrics use simple data inputs: an institution specific compromise vector $C$, and the adjacency matrix of the network graph of financial institution linkages $E$. The risk metrics are general, i.e., independent of the particular definitions of $C$, $E$, and also complement and extend systemic risk measures in the extant literature.

Modeling extensions are also envisaged. In the current version of the model the compromise vector $C$ is independent of the connectivity matrix $E$. Making $C$ a function of $E$ (and vice versa) leads to interesting additional implications, and of course, fresh econometric questions. For example, $C$ may be an increasing function of $E$, but then $E$ may be a decreasing function of $C$, making it unclear as to whether an increase in risk or transaction volume always leads to leads to a higher level of potential systemic risk. Issues such as the structure of the network and the interaction of its components are addressed in the models of Allen, Babus, and Carletti (2012); Glasserman and Young (2013); Elliott, Golub, and Jackson (2014). The welfare implications of over-linking are discussed in the contagion model of Blume, Easley, Kleinberg, Kleinberg, and Tardos (2011).

How to construct composite connectivity matrices across markets is also an interesting issue. One may get a network matrix from transactions in the CDS market (for example see Getmansky, Girardi, and Lewis (2014)) and another from the bond markets, but the question of putting these two matrices (call them $E_1$ and $E_2$) together into one composite $E$ matrix requires a weighting scheme or other collapsing technical condition. One solution to this would be to construct $E$ from bilateral CVA (credit valuation adjustment) numbers, because this directly measures the exposure of each financial institution to another across all products and asset classes. Using counterparty exposures as a device is also considered in the “10-by-10-by-10” systemic risk measurement approach recommended in Duffie (2011).

From a regulatory point of view, there are many applications for this framework. First, Carciente, Kenett, Avakia, Stanley, and Havlin (2015) have an interesting model where a network of banks intersects with a network of asset markets.
imposition of additional capital required may be based on a composite score computed from risk decomposition numbers, taking into account additional informative metrics such as criticality, risk increments, and spillover risk. (See a proposal for this in Espinosa-Vega and Sole (2010).) Second, this composite score may be used to allocate supervision money across various financial institutions. Third, the systemic score can be tracked over time, and empirical work will be needed to backtest whether the systemic score $S$ is a useful early warning predictor of systemic risk events. Using a different approach, Kritzman, Li, Page, and Rigobon (2011); Reyngold, Shmyra, and Stein (2013) find predictability of systemic risk. Fourth, an analysis of network robustness in addition to measuring systemic risk is a complementary analysis, for example Allen and Gale (2000); Callaway, Newman, Strogatz, and Watts (2000).
A Appendix: Other Systemic Risk Measures

This Appendix provides a brief summary of some of the popular systemic risk measures proposed in the literature, and discussed in the paper.

1. Billio, Getmansky, Lo, and Pelizzon (2012) define two measures of systemic risk across banks, hedge funds, broker/dealers, and insurance companies. The idea is to measure correlations among institutions directly and unconditionally using principal components analysis and Granger causality regressions, and thereby assess the degree of connectedness in the financial system.

In their framework, total risk of the system is the variance of the sum of all financial institution returns, denoted $\sigma_S^2$. PCA comprises an eigenvalue decomposition of the covariance matrix of returns of the financial institutions, and systemic risk is higher when the number of principal components $n$ that explain more than a threshold $H$ of the variation in the system is small. Using notation in their paper,

$$h_n = \frac{\omega_n}{\Omega} > H,$$

where $h_n$ is the fraction of $\sigma_S^2$ that is explained by the first $n$ components, i.e., $\Omega = \sum_{i=1}^{N} \lambda_i$ and $\omega_n = \sum_{i=1}^{n} \lambda_i$, where $\lambda_i$ is the $i$-th eigenvalue. We note that $\sigma_S$ is linear homogenous, so can be decomposed to obtain the risk contribution of each financial institution, in the same manner as is done for our network risk measure $S$.

In addition to this covariance matrix based measure of systemic risk, Billio, Getmansky, Lo, and Pelizzon (2012) also create a network using Granger causality. This directed network is represented by an adjacency matrix of values $(0, 1)$ where node $i$ connects to node $j$ if the returns of bank $i$ Granger cause (in a linear or nonlinear way) those of bank $j$, i.e., edge $E_{i,j} = 1$. This adjacency matrix is then used to compute connectedness measures of risk such as number of connections, fraction of connections, centrality, and closeness. These measures correspond to some of those presented in the exposition above, and the first two report an aggregate measure of system-wide risk, different from the $S$ measure developed in this paper. Again, since system-wide risk is defined as a count of the number of connections, it is easy to determine what fraction is ascribable to any single financial firm. They applied the metrics to U.S. financial institution stock return data, and in a follow-up paper, to CDS spread data from U.S., Europe, and Japan (see (Billio, Getmansky, Gray, Lo, Merton, and Pelizzon, 2014)), where the global system is also found to be highly interconnected.
Overall, we note a strong complementarity between the analyses in Billio, Getmansky, Lo, and Pelizzon (2012) and our paper, and using the network matrix in their paper, we may implement our systemic risk score $S$ as well. Hence, this paper extends and uses the results in this earlier work.

2. The $CoVaR$ measure of Adrian and Brunnermeier (2010) estimates a bank or the financial sector’s Value at Risk ($VaR$) given that a particular bank has breached its $VaR$. They use quantile regressions on asset returns ($R$) using data on market equity and book value of debt. Pairwise $CoVaR(j|i)$ for bank $j$ given bank $i$ is at $VaR$ is defined implicitly as the quantile $\alpha$ satisfying

$$Pr[R_j \leq -CoVaR_\alpha(j|i)|R_i = -VaR_\alpha(i)] = \alpha$$

where $VaR(i)$ is also defined implicitly as $Pr[R_i \leq -VaR_\alpha(i)] = \alpha$. The actual measure of systemic risk is then

$$\Delta CoVaR_\alpha(j|i) = CoVaR_\alpha(j|i) - VaR_\alpha(j)$$

The intuition here is one of under-capitalization when a systemic event occurs, i.e., extra capital needed because capital needed for solvency at the time of a systemic event ($CoVaR_\alpha(j|i)$) is greater than capital needed in normal times ($VaR_\alpha(j)$). Replacing $j$ with the system’s value $\Delta CoVaR_\alpha(S|i)$ gives an aggregate measure of systemic risk. However, this is still not an aggregate measure of risk (such as $S$ in this paper), rather one that assesses the systemic risk increment or contribution of the $i$-th financial institution.

3. The AR (Absorption Ratio) metric of Kritzman, Li, Page, and Rigobon (2011) uses another approach to measure systemic risk. It calculates how many eigenvectors are needed to explain the variation in industry returns. The fewer eigenvectors needed, the inference is that there is more systemic risk since the sources of risk are more unified. If the AR is low then it means that the sources of risk are disparate. The AR is computed as follows.

$$AR = \frac{\sum_{i=1}^{n} \sigma_{Ei}^2}{\sum_{j=1}^{N} \sigma_{Aj}^2},$$

where $n$ is the number of eigenvectors used (in their paper Kritzman, Li, Page, and Rigobon (2011) use 1/5 the number of assets ($N$). The variance of the eigenvectors is denoted $\sigma_{Ei}^2$, and that of the assets is $\sigma_{Aj}^2$. Reyngold, Shnyra, and Stein (2013) implement a modified version of the AR ratio by using the covariance matrix of asset
(not equity) returns only for financial firms, where asset values are derived from the a structural credit model. They also only use the first eigenvector’s variance since the data is restricted to a single industry. This measure is denoted as the Credit Absorption Ratio (CAR).

4. The $SES$ (systemic expected shortfall) measure of Acharya, Pedersen, Philippon, and Richardson (2011) captures the amount by which an otherwise appropriately capitalized bank is undercapitalized in the event of a systemic crisis. It is related to $MES$ (marginal expected shortfall), which is the average return of a financial institution for the 5% worst days in the market. Mostly, $SES$ is analogous to $CoVaR$ where Value-at-Risk is replaced with expected shortfall (ES), though the implementation details and variables used differ in the paper of Acharya, et al. We may think of $SES$ as the equity shortfall a firm experiences when aggregate banking equity $e(S)$ is below a threshold $H$, i.e.,

$$SES(j) = E[H(j) - e(j)|e(S) \leq H]$$ (8)

where $H(j)$ is the desired threshold level of equity for bank $j$, with equity level $e(j)$. $SES$ has useful properties in that it is in dollar terms and scales with institution size, so that is it easily aggregated. The DIP (distressed insurance premium) measure of Huang, Zhou, and Zhou (2011) is similar to the SES of Acharya, Pedersen, Philippon, and Richardson (2011) in that it also captures the expected losses of a financial institution conditional on losses being greater than a threshold level.
References


